

CONSTRUCTION OF CONNECTION INDUCING MAPS BETWEEN PRINCIPAL BUNDLES. PART I

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0. Introduction

Consider two C^∞ -smooth principal bundles, say $X \rightarrow V$ and $Y \rightarrow W$, with the same structure group G and with C^∞ connections Γ on X and Δ on Y , respectively. We look for a C^∞ -map $f: V \rightarrow W$ such that the induced bundle $f^*(Y)$ over V with the induced connection $f^*(\Delta)$ is isomorphic to (X, Γ) . This means that f can be covered by (or lifted to) a morphism of bundles, $F: X \rightarrow Y$, inducing Γ from Δ , which is expressed by $F^*(\Delta) = \Gamma$.

0.1. The problem of inducing connections was first studied by Narasimhan and Ramanan [3] who proved that for a given compact Lie group G and an integer $n = 0, 1, \dots$, there exists a (universal) bundle (Y, Δ) over some (classifying) compact manifold W , such that every G -bundle X over an n -dimensional manifold V with an arbitrary C^∞ -connection Γ can be induced by a C^∞ -morphism $F: X \rightarrow Y$. Furthermore, they give a precise description of the universal connection Δ for the unitary and the orthogonal groups. Namely, if $G = U(p)$ they take the Grassmann manifold $\text{Gr}_p(\mathbb{C}^q)$ for W and use the standard connection Δ on the canonical bundle $Y \rightarrow \text{Gr}_p(\mathbb{C}^q)$ (here Y is the Stiefel manifold of orthogonal p -frames in \mathbb{C}^q). The dimension q for which they prove the existence of F is $q = (n + 1)(2n + 1)p^3$, where $n = \dim V$. Similarly, for $G = O(p)$, their method provides a connection inducing map into the real Grassmann manifold $\text{Gr}_p(\mathbb{R}^q)$, again for $q = (n + 1)(2n + 1)p^3$.

0.2. The result by Narasimhan-Ramanan was improved for $G = O(p)$ by Gromov (see 2.2.6 in [1]) who showed the existence of a connection inducing map $f: V \rightarrow \text{Gr}_p(\mathbb{R}^q)$ for $q = \max(p(n + 1), p(n + 2) + n)$. Furthermore, if the manifold V is parallelizable and the bundle $X \rightarrow V$ is trivial, then

$q = p(n + 2)$ suffices for the existence of a map $f: V \rightarrow \text{Gr}_p(\mathbf{R}^q)$ inducing a given connection on X . The improvement was achieved by using the theory of *topological sheaves* (instead of a partition of unity argument used in [3]) to build the (global) map f out of local (connection inducing) maps.

0.3. In this paper, we study connection inducing maps between arbitrary bundles. In particular, in §2 we prove the following:

Theorem. *Let Γ be an arbitrary connection on a trivial $\text{O}(p)$ -bundle over a stably parallelizable manifold V . If $q \geq p(n + 3)/2$, then there exists a connection inducing map $V \rightarrow \text{Gr}_p(\mathbf{R}^q)$.*

0.4. Remarks. (A) One may, in principle, apply the Theorem to a nontrivial bundle X over a nonstably parallelizable manifold V . Namely, take the trivial $2p$ -dimensional bundle $X' \rightarrow V' \supset V$, where V' is a $(2n - 1)$ -dimensional parallelizable manifold which is the total space of the normal bundle of V and where X' contains X as a subbundle. With this, one easily obtains the existence of the inducing connection map to $\text{Gr}_p(\mathbf{R}^q)$ for $q = 2p(n + 1)$. Of course (compare 0.2), this bound on q is too crude and it will be improved in Part II of this paper.

(B) The construction of a connection inducing map F between principal G -bundles X and Y amounts to solving a certain system of $\alpha = \dim V \times \dim G$ partial differential equations imposed on $\beta = \dim W + \dim G$ unknown functions (see 1.1). Therefore (see [2]), for a fixed Δ and for $\alpha > \beta$ a *generic* connection Γ cannot be induced (even locally) from Δ . This means that inducible connections Γ form a meager subset (depending on Δ) in the space of C^∞ connections on V . In particular, if Y is the canonical $\text{O}(p)$ -bundle over $\text{Gr}_p(\mathbf{R}^q)$, then

$$\alpha = \frac{1}{2}np(p - 1), \quad \beta = p(q - p) + \frac{1}{2}p(p - 1).$$

Hence, a generic connection on V cannot be induced from this Y unless $q \geq (p + 1)/2 + n \cdot (p - 1)/2$. This bound on q asymptotically (for $p, n \rightarrow \infty$) agrees with the inequality $q \geq p(n + 3)/2$ in the above Theorem. In fact, the existence of a (local) connection inducing map for a real analytic connection is established in §2 for $q \geq p(n + 1)/2$.

(C) If $q \geq p(n + 1)$, then the P.D.E. system for connection inducing maps $f: V \rightarrow \text{Gr}_p(\mathbf{R}^q)$ can be reduced to an algebraic system (see [2], [1]). But such a reduction hardly is possible for $q \approx pn/2$. Moreover, an appropriate regularity (see Ω -regularity in §1.2) condition on f makes the linearized P.D.E. equations algebraically solvable. This allows us to apply *Nash's implicit function theorem* for local study of such maps f and to use the theory of topological sheaves for obtaining global results. (Compare 2.2, 2.3 in [1]).

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1. General criteria for the existence of connection inducing maps

1.1. Let us consider the (first order) differential operator $\mathcal{D} = \mathcal{D}_\Delta$ which relates to each morphism (i.e. a bundle homomorphism) $F: X \rightarrow Y$ the induced connection $F^*(\Delta)$ on X for a fixed C^∞ -connection Δ on Y . We view morphisms $X \rightarrow Y$ as sections of the bundle $Z \rightarrow V$ associated to the principal bundle $X \rightarrow V$ with the fiber $= Y$ for the action of G on Y . This Z naturally fibers over $V \times W$ with the fiber $X_v \times Y_w/G$ canonically isomorphic to the space of G -equivariant maps $X_v \rightarrow Y_w$. On the other hand, every C^{r+1} -smooth bundle homomorphism $F: X \rightarrow Y$ by definition is given by a C^{r+1} -map $f: V \rightarrow W$ and a family of G -equivariant maps $F_v: X_v \rightarrow Y_{f(v)}$ which are C^{r+1} -smooth in $v \in V$. Thus F becomes a C^{r+1} -section $V \rightarrow Z$ covering the graph $V \rightarrow W$ of f . The range of the operator $F \mapsto F^*(\Delta)$ consists of the space of C^r -connections in X . These are C^r -sections of the fibration $H \rightarrow V$ whose fiber $H_v \subset H$ for $v \in V$ can be described as follows. Denote by X_v^1 the space of 1-jets (or differentials) of germs of sections $V \rightarrow X$ at v . Namely, X_v^1 consists of linear maps $T_v(V) \rightarrow T(X)$ which project to the identity $\text{Id}: T_v(V) \leftarrow$ by the differential (of the projection map) of the fibration $X \rightarrow V$. The group G naturally acts on X_v^1 and the fiber H_v equals X_v^1/G .

Observe that $\dim H_v = \alpha = \dim V \times \dim G$ and $\dim Z_v = \beta = \dim W + \dim G$. Therefore, the connection inducing equation $\mathcal{D}_\Delta(F) = \Gamma$ amounts to α equations in β unknown functions.

Our next objective is to describe an open subset A in the space of 1-jets of germs of sections $V \rightarrow Z$, such that the connection inducing operator $\mathcal{D}_\Delta: F \mapsto F^*(\Delta)$ becomes *infinitesimally invertible* on A . First, recall the pertinent definitions from [1, 2.3.1]. Let \mathcal{D} be a nonlinear first order differential operator between spaces of sections of two arbitrary fibrations Z and H over V . The operator \mathcal{D} (acting from sections of Z to those of H) is called *infinitesimally invertible* on a subset $A \subset Z^1$, where Z^1 stands for the space of 1-jets of germs of sections $V \rightarrow Z$ if for every section $F: V \rightarrow Z$ whose 1-jet sends V to A the linearization of \mathcal{D} at F , called L_F , admits a right inverse M_F which is a linear differential operator. Here we are interested in the case where M_F is a zero order operator which is (nonlinear) differential of order one in F . The resulting operator $M_F(\cdot)$ (in two variables F and \cdot) is called an infinitesimal inversion of order zero and defect (that is the order of M in F) one (see 2.3.1. in [1] for a detailed discussion).

1.2. To describe the pertinent set A for the connection inducing operator $\mathcal{D}(F) = F^*(\Delta)$ we invoke the bundle $\tilde{Y} \rightarrow W$ associated to Y whose fiber is the Lie algebra of G with the adjoint action of G . Denote by $\Omega: T(W) \otimes T(W) \rightarrow \tilde{Y}$ the curvature form of Δ . Call a linear subspace $T' \subset T_w(W)$ for $w \in W$ Ω -regular if one of the following three (obviously) equivalent conditions is satisfied:

(i) For some (and hence for every) basis τ_1, \dots, τ_n in T' the linear system

$$(1) \quad \Omega_w(\tau_i, \partial) = l_i, \quad i = 1, \dots, n,$$

is solvable in $\partial \in T_w(W)$ for every n -tuple of vectors l_i in the Lie algebra \mathfrak{g} of G .

(i)' The homogeneous system

$$(1)' \quad \Omega_w(\tau_i, \partial) = 0, \quad i = 1, \dots, n,$$

is nonsingular. Namely, the dimension of the space of solutions equals $(\dim W - n \dim \mathfrak{g})$.

(ii) The linear map $T_w(W) \rightarrow \text{Hom}(T', \mathfrak{g})$ given by $\tau \mapsto h_\tau(\tau') = \Omega_w(\tau, \tau')$ is surjective.

1.3. Remark. This definition will be used in §2 for subspaces $T' \subset T$, where T is an arbitrary linear space endowed with some bilinear vector-valued form Ω .

1.4. Example. If Ω is an ordinary (i.e. \mathbf{R} -valued) form, then $T' \subset T$ is Ω -regular if and only if $T' \cap \ker \Omega = 0$, where $\ker \Omega = \{t \in T \mid \Omega(t, t') = 0 \text{ for all } t' \in T\}$. In particular, if Ω is nonsingular (symplectic) then every subspace in T is Ω -regular (compare [1, 3.4]). Notice that the curvature Ω of the canonical Ω of the canonical $O(2)$ -bundle over the Grassmannian manifold $\text{Gr}_p(\mathbf{R}^q)$ is symplectic (this Ω can be regarded as an \mathbf{R} -form since the Lie algebra of $O(2)$ is $\approx \mathbf{R}$).

1.5. Take a linear map $\varphi: T_v(V) \rightarrow T_w(W)$ and let a 1-jet $\Phi \in Z^1$ lie over φ . (If Φ is the 1-jet $J_F^1(v)$ for a morphism $F: X \rightarrow Y$, then φ is the differential D_f of the underlying map $f: V \rightarrow W$ at $v \in V$.) Call φ Ω -regular if it is injective and if the image $\varphi(T_v(V)) \subset T_w(W)$ is Ω -regular for all $v \in V$. Call Φ Ω -regular if the underlying map φ is Ω -regular. Then define the subset $A \subset Z^1$ as the set of the Ω -regular 1-jets Φ . According to this terminology, we say that $F: X \rightarrow Y$ (as well as the underlying map $f: V \rightarrow W$) is Ω -regular if the 1-jet $J_F^1: V \rightarrow Z^1$ sends V into A . This is equivalent to the Ω -regularity of the differential $D_f: T(V) \rightarrow T(W)$ at every point $v \in V$.

1.6. Proposition. *The connection inducing operator $\mathcal{D}: F \mapsto F^*(\Delta)$ on Ω -regular morphisms F admits an infinitesimal inversion M of order zero and defect one.*

Proof. First, we must linearize the operator \mathcal{D} at some morphism F . To do this, we take a smooth 1-parametric family of morphisms $F_t: X \rightarrow Y$ for $t \in [0, 1]$, such that $F_0 = F$ and study the family of the induced connection $\Gamma_t = \mathcal{D}(F_t)$. The derivative $\frac{d}{dt}\Gamma_t$ is a 1-form on V with the values in the vector bundle \tilde{X} induced from \tilde{Y} by the map $f_0 = f: V \rightarrow W$ corresponding to $F_0 = F$. Let us express this form in terms of Ω . Let $V' = V \times [0, 1]$ and $X' = X \times [0, 1] \rightarrow V'$. Consider the connection Δ' on X' induced by the morphism $X' \rightarrow Y$ defined by $(x, t) \rightarrow F_t(x)$. Denote by ∂ the field $\partial/\partial t$ on $V' = V \times [0, 1]$ for $t \in [0, 1]$, let $\tilde{\delta}$ be the corresponding field on X' , and let $\tilde{\delta}^\vee$ be the Δ' -vertical component of $\tilde{\delta}$. Then the value of the 1-form $\Gamma'_t \stackrel{\text{def}}{=} \frac{d}{dt}(\Gamma_t)$ on every $\tau \in T(V = V \times t)$ is given by

$$(2) \quad \Gamma'_t(\tau) = \Omega'(\tau, \partial) + d\tilde{\delta}^\vee(\tau),$$

where d stands for the Δ' -horizontal differential, and Ω' is the curvature of Δ' . Now let us denote by L_F the linearization of \mathcal{D} at F and, assuming the map f is Ω -regular, let us resolve the linearized equation

$$(3) \quad L_F(\tilde{\delta}) = l,$$

where l is a given section $V \rightarrow \tilde{X}$ and $\tilde{\delta}$ is the unknown infinitesimal deformation (vector field) of F . We shall seek a solution of (3) among Δ' -horizontal fields $\tilde{\delta}$. In terms of the connection Δ' , this *horizontality* is expressed (with a slight abuse of notations) by

$$(4) \quad \tilde{\delta}^\vee = 0.$$

Next we introduce another linear algebraic equation (or rather a system of equations) for the projection ∂ of $\tilde{\delta}$ to $T(V \times [0, 1])$,

$$(5) \quad \Omega'_0(\tau, \partial) = l$$

for all $\tau \in T(V_0 = V \times 0)$, where $\Omega'_0 = \Omega'|_{V \times 0}$. According to (2) every $\tilde{\delta}$ satisfying (4) and (5) is a solution of (3). Since f is Ω -regular, the relation (5) can be expressed, at every point $v \in V_0$, by the following *nonsingular* system of linear *algebraic* equations:

$$(6) \quad \Omega'_0(\tau_i, \partial) = l(\tau_i), \quad i = 1, \dots, n,$$

for a fixed basis τ_1, \dots, τ_n in $T_v(V_0)$. Hence, solutions of (4) and (5) form an affine (sub)-space of dimension $d = \dim W - n \dim g$ and the solutions of (3) are sections of a d -dimensional affine (sub)-bundle over V_0 . Such a bundle always admits a section. Moreover, one can easily choose a specific section, say ∂_0 , with an appropriate partition of unity or with an auxiliary Riemannian metric in the ambient vector bundle (see [1, 2.31]). Finally, we define the infinitesimal inversion $M = M_F$ of \mathcal{D} by $M_F(l) = \partial_0$ which, according to our construction, satisfies the desired (infinitesimal invertibility) relation $L_F(M_F(l)) = l$.

1.7. Now, by specializing analytic results in [1, 2.3.2] to our \mathcal{D} , we obtain the following

Corollaries. Denote by $\{\Gamma\}^r$ the space of connections Γ on X with the fine C^r -topology (if V is compact, this is the ordinary C^r -topology).

1.8. **Corollary.** For every Ω -regular C^∞ -morphism $F: X \rightarrow Y$ there exists a neighborhood $\mathcal{U} \subset \{\Gamma\}^2$ of the induced connections $\mathcal{D}(F) \in \{\Gamma\}^\infty \subset \{\Gamma\}^2$, such that every C^r -connection $\Gamma' \in \mathcal{U}$ for $r \geq 2$ can be induced by a C^r -morphism $F': X \rightarrow Y$. Moreover, if Δ and Γ' are real analytic, then F' also can be chosen real analytic.

1.9. **Corollary.** Suppose, for some $v \in V$, there exists an Ω -regular homomorphism $\varphi: T_v(V) \rightarrow T_w(W)$ for some $w \in W$, and a Lie algebra isomorphism $\tilde{\Phi}: \tilde{X}_v \rightarrow \tilde{Y}_w$ such that the curvature Ω of Δ at w induces the curvature form Ω'_v of a given C^r -connection Γ on X . That is,

$$(7) \quad \tilde{\Phi}(\Omega'_v(\tau_1, \tau_2)) = \Omega_w(\varphi(\tau_1), \varphi(\tau_2))$$

for all τ_1 and τ_2 in $T_v(V)$. Then for $r \geq 2$ the connection Γ near $v \in V$ can be induced from Δ . Namely, there exists a neighborhood $\mathcal{U} \subset V$ of v such that the connection Γ over \mathcal{U} can be induced by a C^r -morphism F of the bundle X (now restricted to \mathcal{U}) to Y . Moreover, one may choose F such that the differential of the underlying map f satisfies $Df|_{T_v(V)} = \varphi$.

1.10. The following global version of 1.9 follows from the theory of flexible sheaves (see 2.2 in [1]). Consider a continuous morphism $\Phi: X \rightarrow Y$ and let $\varphi: T(V) \rightarrow T(W)$ be a fiberwise injective homomorphism whose underlying map $V \rightarrow W$ equals that of Φ . Denote by $\tilde{\Phi}: \tilde{X} \rightarrow \tilde{Y}$ the fiberwise Lie algebra isomorphism associated to Φ and let

$$(8) \quad (\varphi, \tilde{\Phi})^*(\Omega) = \Omega',$$

where Ω is the curvature of Δ and Ω' is the curvature of a given C^r -connection Γ in X . (The relation (8) means that

$$\tilde{\Phi}\Omega'(\tau_1, \tau_2) = \Omega(\varphi(\tau_1), \varphi(\tau_2))$$

for all tangent fields τ_1 and τ_2 on V .)

If the homomorphism φ is Ω -regular at all $v \in V$ and if $r \geq 2$ (recall that the connection Δ in Y is C^∞) then, under the following condition (*) there exists a C^r -morphism $F: X \rightarrow Y$ such that $F^*(\Delta) = \Gamma$.

(*) There exists a G -bundle \bar{X} over some manifold \bar{V} with a C^r -connection $\bar{\Gamma}$ and a morphism $P: X \rightarrow \bar{X}$, such that:

(a) $P^*(\bar{\Gamma}) = \Gamma$,

(b) the underlying map $p: V \rightarrow \bar{V}$ is a submersion such that the pull-back $p^{-1}(\bar{v})$ is an open manifold of positive dimension for all $\bar{v} \in \bar{V}$. (Recall a

manifold is open if it contains no component which is a compact manifold without boundary.)

1.11. Remark. Condition (*) may seem rather restrictive. However, it can be applied to any V and Γ as follows. Take $V' = V \times \mathbf{R}$ with the obvious projection $p: V' \rightarrow V$ and with the induced connection Γ' on $X' = X \times \mathbf{R} \rightarrow V'$. Then Γ' does satisfy (*). On the other hand, $\Gamma'|_{V \times 0} = \Gamma$. Thus, by inducing Γ' from Δ we also induce Γ from Δ .

1.12. In order to apply (1.10) and (1.11) let us state the following algebraic Lemma which follows from 2.3.

Lemma. Let $\varphi^0: T(V) \rightarrow T(W)$ be a continuous Ω -isotropic homomorphism (which means $\Omega|\varphi^0(T_v(V)) = 0$ for all $v \in V$), such that the bundle induced from Y by the continuous map $\psi: V \rightarrow V \times W$ underlying φ^0 is isomorphic to X . If φ^0 is Ω -regular then, for any arbitrary 2-form $\Omega': T(V) \otimes T(V) \rightarrow \tilde{X}$, there exists an Ω -regular homomorphism $\varphi: T(V) \rightarrow T(W)$ and a morphism $\Phi: X \rightarrow Y$, both lying over ψ , such that $(\varphi, \Phi)^*(\Omega) = \Omega'$.

1.13. Corollary. Denote by T' the Whitney sum of $T(V)$ with the trivial line bundle and let $\varphi^0: T' \rightarrow T(W)$ be an Ω -regular and Ω -isotropic homomorphism, such that the bundle induced from Y by the underlying continuous map $V \rightarrow W$ is isomorphic to X . Then an arbitrary C^r -connection Γ on X for $r \geq 2$ can be induced by a C^r -morphism $F: X \rightarrow Y$.

Proof. Compose φ^0 with the obvious (fiberwise injective) projection $T(V \times \mathbf{R}) \rightarrow T'$ and apply the Lemma to the resulting Ω -regular and Ω -isotropic homomorphism $T(V \times \mathbf{R}) \rightarrow T(W)$. Thus we get an Ω -regular homomorphism $\varphi': T(V \times \mathbf{R}) \rightarrow T(W)$ and a morphism $\Phi': X' \rightarrow Y$ for $X' = X \times \mathbf{R} \rightarrow V \times \mathbf{R}$ such that $(\varphi', \Phi')^*(\Omega) = \Omega'$ for the curvature form Ω' of the connection Γ' induced from Γ by the projection $V \times \mathbf{R} \rightarrow V$. Now, by applying 1.10, we induce Γ' from Δ and, by restricting Γ' to $V = V \times 0$ (compare 1.11) we induce Γ as well.

1.14. Let us specialize 1.13 to the case of the *trivial* vector bundle $X = G \times V \rightarrow V$ over a stably parallelizable manifold V . Assume, there is an $(n + 1)$ -dimensional Ω -regular and Ω -isotropic subspace $T'_w \subset T_w(W)$ for some $w \in W$, where the Ω -isotropy means $\Omega(\tau'_1, \tau'_2) = 0$ for all τ'_1, τ'_2 in T' . Then the required φ^0 does exist. Indeed, since the bundle T' is trivial it can be induced by a (constant) map $\psi_w: V \rightarrow w \in W$ from the subspace T'_w viewed as a bundle over $\{w\}$ and since X is trivial it is isomorphic to $\psi_w^*(Y)$. Thus we conclude:

1.15. Proposition. Let Γ be a C^r -connection in a trivial bundle X over a stably parallelizable n -dimensional manifold V . If the connection Δ in Y admits an $(n + 1)$ -dimensional Ω -regular and Ω -isotropic subspace in some tangent space $T_w(W)$ then, for $r \geq 2$, there exists a C^r -morphism $X \rightarrow Y$ which induces Γ from Δ .

1.16. Remark (compare 1.9). Extend a connection Γ on X to a connection Γ' on $X' = X \times \mathbf{R} \rightarrow V' = V \times \mathbf{R}$ and take a Γ -inducing Ω -regular morphism F of $X = X \times 0$ to Y . Let us try to extend F to a Γ' -inducing morphism $X' \rightarrow Y$. Denote by ∂^h the Γ' -horizontal lift to X' of the field $\partial = \partial/\partial t$ on V' and observe that the equation $(F')^*(\Delta) = \Gamma'$ implies (compare (2))

$$(9) \quad \Omega\left(\frac{\partial f'}{\partial t}, \partial_\tau f'\right) = \tilde{F}'\Omega'(\partial, \tau)$$

for all tangent fields τ on V' , where the following notations are used:

Ω' is the curvature of Γ' ,

\tilde{F}' is the associated homomorphism of the Lie algebra bundle, $\tilde{F}': \tilde{X}' \rightarrow \tilde{Y}$, $\partial f'/\partial t$ and $\partial_\tau f'$ are the images of the fields $\partial/\partial t$ and τ correspondingly under the differential of the map $f': V \rightarrow W$ underlying F' .

If $F'|_X = F$, then the Ω -regularity of F allows one to resolve equation (9) in $\partial f'/\partial t$ and to bring it to the evolution (or Cauchy-Kowalewsky) form. We write it as

$$(10) \quad \frac{\partial f'}{\partial t} = \Omega^{-1}(\partial_\tau f', \tilde{F}'\Omega'(\partial, \tau)),$$

where the "inverse" Ω^{-1} is defined at (v, t) in so far as $f|_{V \times t}$ is Ω -regular at (v, t) .

Now, one can easily see that

1.17. Lemma. *If a map F' satisfies equation (10) (which, whenever defined, is equivalent to (9)) and the conditions (a) $F'|_X = F$ and (b) the differential of F' sends ∂^h to a Δ -horizontal field, then $F'^*(\Delta) = \Gamma'$.*

The proof follows by reversing the computation which brought up equation (9). The same conclusion equally applies to small neighborhoods $\mathcal{U}' \subset V \times \mathbf{R}$ of $V \times 0$. Therefore, the extension of F to \mathcal{U}' reduces to solving the evolution system expressed by (10) and the above condition (b) with the initial data (a).

1.18. Corollary. *If the connections Γ' and Δ and the morphism F are real analytic, then for some neighborhood \mathcal{U}' of $V \times 0$ in $V \times \mathbf{R}$, there exists a real analytic morphism of $X'|_{\mathcal{U}'}$ to Y which induces Γ' on $X'|_{\mathcal{U}'}$, where $X'|_{\mathcal{U}'}$ denotes the restriction of X' to \mathcal{U}' .*

Proof. Apply the Cauchy-Kowalewsky theorem (compare [2]).

Now let us return to the assumptions of Corollary 1.9 where we had an Ω -regular linear map $\varphi: T_v(V) \rightarrow T_w(W)$ inducing the curvature of Γ via some $\tilde{\Phi}$. In the real analytic case, the above Corollary 1.18 insures the existence of some neighborhood \mathcal{U}' of $(v, 0) \in V \times \mathbf{R}$ and of a Γ' -inducing morphism of $X'|_{\mathcal{U}'}$ to Y for every (real analytic!) connection Γ' on $V \times \mathbf{R}$. (This result, as was told to me by Professor M. Gromov, is due to E. Cartan.)

2. Ω -regular subspaces

2.1. Consider linear spaces T, T' , and g and let Ω be an antisymmetric bilinear form $T \otimes T \rightarrow G$. For every homomorphism $\varphi: T' \rightarrow T$ let $\delta(\varphi)$ denote the induced form $\varphi^*(\Omega)$ on T' . Recall that a homomorphism φ is called Ω -regular if the linear map $T \rightarrow \text{Hom}(T', g)$ given by $(\tau \mapsto h_\tau(\tau') = \Omega(\tau, \varphi(\tau')))$ is surjective. Restrict the above map $\varphi \mapsto \delta(\varphi)$ to the space of Ω -regular homomorphisms $T' \rightarrow T$.

Lemma. *The map*

$$\delta: \text{Reg Hom}(T', T) \rightarrow (\text{the space of antisymmetric 2-forms } T' \otimes T' \rightarrow g)$$

is a submersion.

Proof. Fix a basis $\tau'_i \in T'$, $i = 1, \dots, n$, and let $\tau_i = \varphi(\tau'_i)$. The surjectivity of the differential of δ at φ is equivalent to solvability in $x_i \in T$, $i = 1, \dots$, of the linear system

$$(11) \quad \lim_{\epsilon \rightarrow 0} \epsilon^{-1} [\Omega(\tau_i + \epsilon x_i, \tau_j + \epsilon x_j) - \Omega(\tau_i, \tau_j)] = \omega_{ij}$$

for any given antisymmetric matrix of vectors $\omega_{ij} \in g$, $1 \leq i, j \leq n$. If φ is Ω -regular, one can solve for every fixed j the system

$$\Omega(\tau_i, x_j) = \frac{1}{2} \omega_{ij}, \quad i = 1, \dots, n.$$

Since Ω and ω are antisymmetric, these solution x_j satisfy the system of equations

$$\Omega(\tau_i, x_j) + \Omega(x_i, \tau_j) = \omega_{ij}, \quad 1 \leq i, j \leq n,$$

which is equivalent to (11).

2.2. Corollary. *Consider vector bundles T, T' , and g over V and a bilinear antisymmetric form $\Omega: T \otimes T \rightarrow g$. If $\varphi: T' \rightarrow T$ is an Ω -regular (continuous) homomorphism, then for every sufficiently small antisymmetric form ω on T' there exists an Ω -regular homomorphism $\varphi': T' \rightarrow T$ such that $(\varphi')^*(\Omega) = \varphi^*(\Omega) + \omega$.*

(If V is noncompact, "small" refers to the fine C^0 -topology in the space of sections $\text{Hom}(T' \otimes T' \rightarrow g)$.)

Proof. The map δ which now sends (the total space of) the bundle $\text{Hom}(T' \rightarrow T)$ to the bundle of g -valued forms on T' , is a topological submersion on the subset $\text{Reg Hom}(T', T) \subset \text{Hom}(T', T)$ because it is a submersion over every point $v \in V$ by the Lemma in 2.1.

2.3. Corollary. *Let T, T', g and Ω be as in 2.2 and let $\varphi^0: T' \rightarrow T$ be an Ω -regular and Ω -isotropic homomorphism. That is, $(\varphi^0)^*(\Omega) \equiv 0$. Then, every g -valued 2-form ω on T' can be induced from Ω by an Ω -regular homomorphism $\varphi: T' \rightarrow T$.*

Proof. By 2.2 there exists a small positive function ε on V and an Ω -regular homomorphism φ' , such that $(\varphi')^*(\Omega) = \varepsilon\omega$. Then the homomorphism $\varepsilon^{-2}\varphi'$ induces ω .

2.4. In order to make sure that the general results in §1 are nonvacuous, we need examples of connections Δ on W such that the tangent bundle $T(W)$ contains sufficiently many Ω -regular subspaces.

2.5. Lemma. *Let T and g be linear spaces and let T' be a subspace in T . Set $m = \dim T$, $n = \dim T'$, and $k = \dim g$. Then, in the following three cases (and, as one can easily see, only in these cases) there exists a g -valued 2-form Ω on T (i.e. an antisymmetric bilinear map $T \otimes T \rightarrow g$) for which T' is Ω -regular.*

- (i) $n = 1$, $m > k$.
- (ii) $k = 1$ and m is even.
- (iii) $m \geq n \cdot k$ and $m > n$.

Proof. Take a subspace $S \subset T$ complementary to T' . Then, in case (i) take any surjective linear map $\Omega_0: S \otimes T' \rightarrow g$ and define Ω by

$$\Omega(s_1 + t'_1, s_2 + t'_2) = \Omega_0(s_1 \otimes t'_2) - \Omega_0(s_2 \otimes t'_1)$$

for all $s_1, s_2 \in S$ and $t'_1, t'_2 \in T'$.

Next, let $\dim g = 1$ and $\dim T$ be even. Then take any nonsingular \mathbf{R} -valued form on T for Ω . This concludes case (ii). Furthermore, if $\dim T' < \dim T$ this applies to an even dimensional subspace $T_0 \supset T'$ in T and thus yields case (iii) for $\dim g = 1$. Now, let $\dim g \geq 2$ and $\dim T' \geq 2$. Then there exists a g -valued 2-form Ω' on T' for which the homomorphism $h': T' \rightarrow \text{Hom}(T', g)$ given by $t' \rightarrow h'_t(t'') = \Omega(t', t'')$ is injective. Indeed, take $\Omega' = (\omega_1, \omega_2, \dots, \omega_k)$ for $k = \dim g$ where the \mathbf{R} -valued forms ω_1 and ω_2 have ranks $\geq \dim T' - 1$ and $\ker \omega_1 \cap \ker \omega_2 = 0$, where, by definition,

$$\ker \omega = \{t' \in T' \mid \omega(t', t'') = 0 \text{ for all } t'' \in T'\}.$$

If $\dim T \geq \dim T' \cdot \dim g$, then there exists a linear map $h: S \rightarrow \text{Hom}(T', g)$ such that the images $h(S)$ and $h'(T)$ span $\text{Hom}(T', g)$. Finally, we define

$$\Omega(s_1 + t'_1, s_2 + t'_2) = \Omega'(t'_1, t'_2) + h_{s_1}(t'_2) - h_{s_2}(t'_1).$$

2.6. Remark. If g is the Lie algebra of G and T is the tangent space of a manifold W at a point $w \in W$, then for any g -valued 2-form Ω_0 on T there obviously exists a C^∞ (and even real analytic) connection in any given G -bundle $Y = G \times W \rightarrow W$ whose curvature at w equals Ω_0 .

2.7. Let us return to the G -bundles $X \rightarrow V$ and $Y \rightarrow W$ where $\dim G = k$, $\dim V = n$, and $\dim W = m$, and assume the bundle X to be trivial. We

combine the results in §1 with 2.5 and 2.6 and obtain the following

Corollary. *If the dimensions k , m , and n satisfy one of the above conditions (i), (ii), or (iii), then there exist subsets $\{\Gamma\}_0$ and $\{\Delta\}_0$ in the space of connections on X and on Y corresponding which satisfy the following conditions:*

- (1) *The subsets $\{\Gamma\}_0$ and $\{\Delta\}_0$ are nonempty.*
- (2) *The subset $\{\Gamma\}_0$ is open in the fine C^2 -topology in the space of connections in X .*
- (3) *The subset $\{\Delta\}_0$ is open in the fine C^1 -topology in the space of connections in Y .*
- (4) *For every C^∞ -connection $\Delta \in \{\Delta\}_0$ and every C^r -connection $\Gamma \in \{\Gamma\}_0$ for $r \geq 2$ there exists a C^r -morphism $F: X \rightarrow Y$ such that $F^*(\Delta) = \Gamma$.*

(5) *Let $\Delta \in \{\Delta\}_0$ and $\Gamma \in \{\Gamma\}_0$ be real analytic and let Γ' be a real analytic connection in the bundle $X' = X \times \mathbf{R} \rightarrow V \times \mathbf{R}$ such that $\Gamma'|_V = V \times 0$ equals Γ . Then, there exists a neighborhood $\mathcal{U} \subset X \times \mathbf{R}$ of $X \times 0$ such that the connection Γ' over \mathcal{U} can be induced from Δ by a C^{an} -morphism of $X'|_{\mathcal{U}}$ to Y .*

2.8. Example. Let $k = \dim g \geq 2$ and $n = \dim V \geq 0$. Then the above applies for $m \geq kn$, for $m = \dim W$. In particular, we obtain a nonempty open set of real analytic connections Γ' on $V \times \mathbf{R}$ which can be locally induced from some fixed Δ . By Remark (B) in 0.4, such a Δ does not exist for $m < kn$.

2.9. Lemma. *Let T , g , and $T' \subset T$ be linear spaces of dimension m , k , and n (compare 2.5). If*

$$(12) \quad m \geq n(k + 1),$$

then there exists a g -valued 2-form Ω on T for which $T' \subset T$ is Ω -regular and Ω -isotropic.

Proof. Take $S \subset T$ as in the proof of 2.5 and a surjective linear map $h: S \rightarrow \text{Hom}(T', g)$. Then the form $\Omega(s_1 + t'_1, s_2 + t'_2) = h_{s_1}(t'_2) - h_{s_2}(t'_1)$ is the required one.

2.10. Remarks. (a) Inequality (12) obviously is the best possible.

(b) Every sufficiently small perturbation Ω' on Ω admits an n -dimensional Ω -regular and Ω -isotropic subspace $T'' = T''(\Omega') \subset T$.

2.11. Now, as in 2.7, we obtain, with 1.15, the following:

Corollary. *Let the manifold V be stably parallelizable, let $X = V \times G \rightarrow V$ be the trivial bundle and let $Y \rightarrow W$ be an arbitrary G -bundle. If $\dim W \geq (\dim V + 1)(\dim g + 1)$, then there exists a C^∞ -connection Δ on Y such that every C^r -connection Γ on V for $r \geq 2$ can be induced from Δ by a C^r -morphism $F: X \rightarrow Y$. Furthermore, every small C^1 -perturbation of Δ has the same property.*

2.12. Let us turn to the canonical $O(p)$ -bundle Y over the Grassmann manifold $\text{Gr}_p(\mathbf{R}^q)$ with the standard ($O(q)$ -invariant) connection Δ . At every point $w \in \text{Gr}_p(\mathbf{R}^q)$, the tangent space $T_w(\text{Gr}_p(\mathbf{R}^q))$ is identified with the space

$T = \text{Hom}(\mathbf{R}^p, \mathbf{R}^{q-p})$, the Lie algebra \mathfrak{g} of $O(p)$ is represented by antisymmetric $(p \times p)$ -matrices, and the curvature form Ω of the connection Δ is given by the formula (see [1, 3.2.1])

$$(13) \quad \Omega(A_1, A_2) = A_1' A_2 - A_2' A_1$$

for $A_1, A_2 \in T$ where the prime denotes transposition of matrices. Observe that $\dim T = p(q-p)$ and $\dim \mathfrak{g} = p(p-1)/2$.

2.13. Lemma. *Let $\Omega: T \otimes T \rightarrow \mathfrak{g}$ be the \mathfrak{g} -valued 2-form defined by (13) where T is the space of $p \times q'$ matrices for $q' = q-p$ and \mathfrak{g} is the space of antisymmetric matrices of order p . Then, there exists an Ω -regular and Ω -isotropic subspace $T' \subset T$ of dimension $n = 2 \text{ent } q'/p$.*

Proof. Start with the case $p = q'$ and consider the 2-dimensional space generated by the matrices

$$I = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}, \quad I_\lambda = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ 0 & & & & \lambda_p \end{pmatrix}.$$

Since $\Omega(I, I_\lambda) = II_\lambda - I_\lambda I = 0$, this space is Ω -isotropic. It is also Ω -regular, if

$$(14) \quad \lambda_i \neq \lambda_j, \quad \text{for } i \neq j.$$

To see this, look at the linear homogeneous system

$$(15) \quad \Omega(X, I) = 0, \quad \Omega(X, I_\lambda) = 0$$

in the unknown $X \in T$, $T = \text{Hom}(\mathbf{R}^p, \mathbf{R}^p)$, and show the space of solutions to have dimension $\leq p = p^2 - p(p-1)$, where p^2 is the number of unknowns in (15) and $p(p-1)$ is the number of equations. The equation $\Omega(X, I) = 0$ amounts to $X = X'$ (i.e. the matrix X is symmetric) and the equation $\Omega(X, I_\lambda) = X'I_\lambda - I_\lambda'X = I_\lambda X - XI_\lambda = 0$ implies that every element of X , say x_{ij} , satisfies $(\lambda_i - \lambda_j)x_{ij} = 0$. Hence, every solution of (15) is a diagonal matrix and the proof is concluded for $p = q'$. Now, for $q' = sp + p'$ for $0 \leq p' < p$, we take:

$$I^1 = (I, 0, \dots, 0, 0'), \quad I^2 = (0, I, 0, \dots, 0, 0'), \dots, \quad I^s = (0, \dots, 0, I, 0'),$$

$$I_\lambda^1 = (I_\lambda, 0, \dots, 0, 0'), \quad I_\lambda^2 = (0, I_\lambda, 0, \dots, 0, 0'), \dots, \quad I_\lambda^s = (0, \dots, 0, I_\lambda, 0'),$$

where the zeros stand for the zero $(p \times p)$ -matrices and $0'$ is the zero $(p \times p')$ -matrix. The $\text{Span}(I^i, I_\lambda^i)$ (of dimension $2s = 2 \text{ent } q'/p$) is obviously Ω -isotropic and, if condition (14) holds, it is also Ω -regular. Indeed, the system

$$(16) \quad \begin{aligned} \Omega(X, I^i) &= 0, & i &= 1, \dots, s, \\ \Omega(X, I_\lambda^i) &= 0, & i &= 1, \dots, s \end{aligned}$$

(which has $sp(p-1)$ equations in pq' unknowns $X \in T$, $T = \text{Hom}(\mathbf{R}^p, \mathbf{R}^{q'})$), obviously divides into s independent subsystems of the type (15). Hence, by the above, the space is solutions of (16) is $(sp + pp')$ -dimensional.

2.14. Now we can prove the results stated in 0.3. and in the last remark in (B) of 0.4. Let X be a trivial $O(p)$ -bundle over a stably parallelizable manifold V . Then the existence of the Ω -isotropic and Ω -regular subspace in $T(\text{Gr}_p(\mathbf{R}^q))$ established above and Proposition 1.15 insure the existence of the connection inducing map $f: V \rightarrow \text{Gr}_p(\mathbf{R}^q)$ for all $q \geq p(n+3)/2$. Furthermore, in the real analytic case, we apply 1.18 to a small tubular neighborhood $\mathcal{U} \subset V$ of a hypersurface $V_0 \subset V$ and obtain a connection inducing map $f: \mathcal{U} \rightarrow \text{Gr}_p(\mathbf{R}^q)$ for all $q \geq p(n+1)/2$.

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